3. Differential Geometry of Surfaces

3.1 Tangent plane and surface normal

Let us consider a curve $u = u(t)$, $v = v(t)$ in the parametric domain of a parametric surface $\mathbf{r} = \mathbf{r}(u, v)$ as shown in Fig. 3.1. Then $\mathbf{r} = \mathbf{r}(t) = \mathbf{r}(u(t), v(t))$ is a parametric curve lying on the surface $\mathbf{r} = \mathbf{r}(u, v)$. The tangent vector to the curve on the surface is evaluated by differentiating $\mathbf{r}(t)$ with respect to the parameter t using the chain rule and is given by

$$\dot{\mathbf{r}}(t) = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v} \ , \tag{3.1}$$

where subscripts u and v denote partial differentiation with respect to u and v, respectively. The *tangent plane* at point P can be considered as a union

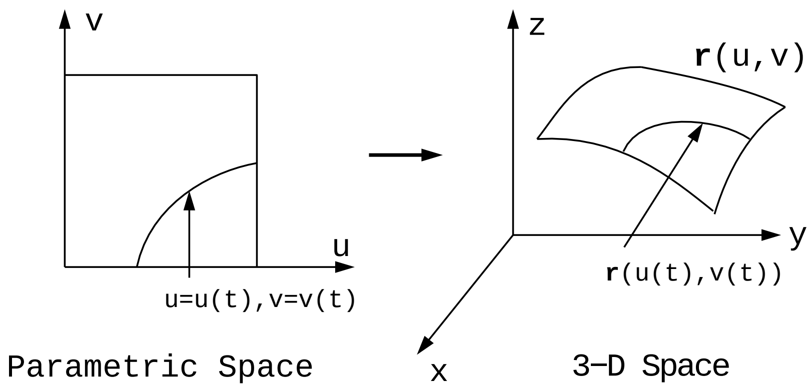

Fig. 3.1. The mapping of a curve in 2-D parametric space onto a 3-D parametric surface

of the tangent vectors of the form (3.1) for all $\mathbf{r}(t)$ through P as illustrated in Fig. 3.2. Point P corresponds to parameters u_p, v_p. Since the tangent vector (3.1) consists of a linear combination of two surface tangents along iso-parametric curves $\mathbf{r}_u$ and $\mathbf{r}_v$, the equation of the tangent plane at $\mathbf{r}(u_p, v_p)$ in parametric form with parameters μ, ν is given by

$$\mathbf{T}_p(\mu,\nu) = \mathbf{r}(u_p,v_p) + \mu\mathbf{r}_u(u_p,v_p) + \nu\mathbf{r}_v(u_p,v_p) \; . \tag{3.2}$$

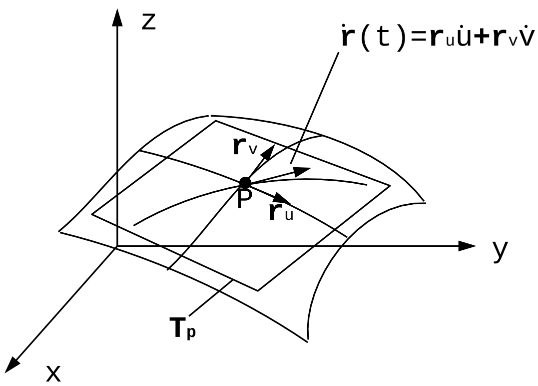

Fig. 3.2. The tangent plane at a point on a surface

The *surface normal vector* is perpendicular to the tangent plane (see Fig. 3.3) and hence the unit normal vector is given by

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \; . \tag{3.3}$$

By using (3.3), the equation of the tangent plane at $\mathbf{r}(u_p, v_p)$ can be written in the implicit form as

$$(\mathbf{r} - \mathbf{r}(u_p,v_p)) \cdot \mathbf{N}(u_p,v_p) = 0 \; , \tag{3.4}$$

where $\mathbf{r}$ is a point on the tangent plane.

Definition 3.1.1. *A regular (ordinary) point P on a parametric surface is defined as a point where $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$. A point which is not a regular point is called a singular point.*

The condition $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ requires that at point P the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ do not vanish and have different directions, i.e. $\mathbf{r}_u$ and $\mathbf{r}_v$ are linearly independent. As we discussed in Sect. 1.3.6, in some design problems we need to employ triangular patches defined by parametrization over a rectangular domain. Such a degenerated patch can be generated by collapsing one boundary curve into a single point or by arranging for two partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$ at one of the corners of a quadrilateral patch to be collinear. In both cases $\mathbf{r}_u \times \mathbf{r}_v$ has zero magnitude at the degenerate corner point and (3.3) cannot be used. Conditions for the existence of surface normals at these degenerate

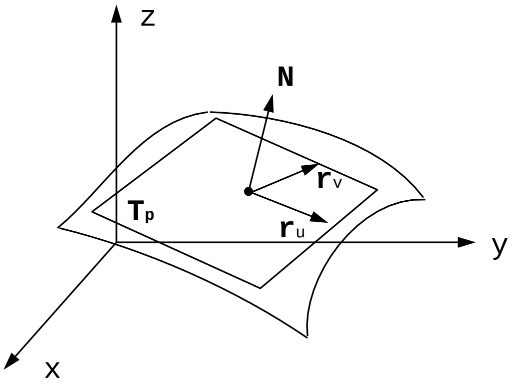

Fig. 3.3. The normal to the point on a surface

corner points have been discussed in [116, 92, 453, 457]. The concept of a regular surface requires additional conditions beyond the existence of a tangent plane everywhere on the surface, such as absence of self-intersections. This concept is presented fully in do Carmo [76].

There are *essential* and *artificial* singularities [444]. The essential singularities arise from specific features of the surface geometry such as the apex of a cone. The artificial singularities arise from the choice of parametrization.

Example 3.1.1. The elliptic cone can be described in a parametric form $\mathbf{r} = (at\cos\theta, bt\sin\theta, ct)^T$, where $0 \le \theta \le 2\pi$, $0 \le t \le l$ and a, b, c are constants. We have

$$\mathbf{r}_\theta = (-at\sin\theta, bt\cos\theta, 0)^T, \quad \mathbf{r}_t = (a\cos\theta, b\sin\theta, c)^T ,$$

thus

$$\begin{aligned} |\mathbf{r}_\theta \times \mathbf{r}_t| &= |bct\cos\theta\mathbf{e}_x + act\sin\theta\mathbf{e}_y - abt\mathbf{e}_z| \\ &= \sqrt{t^2(b^2c^2\cos^2\theta + a^2c^2\sin^2\theta + a^2b^2)} . \end{aligned}$$

We can easily observe that the surface becomes singular only at $t = 0$, which corresponds to the apex of the cone.

The unit normal vector for an implicit surface can be derived by considering two parametric curves $\mathbf{r}_1 = (x_1(t_1), y_1(t_1), z_1(t_1))^T$, $\mathbf{r}_2 = (x_2(t_2), y_2(t_2), z_2(t_2))^T$ lying on an implicit surface $f(x, y, z) = 0$, and intersecting at point P on the surface with different tangent directions. Thus we have the relations:

$$f(x_1(t_1), y_1(t_1), z(t_1)) = 0, \quad f(x_2(t_2), y_2(t_2), z(t_2)) = 0 . \tag{3.5}$$

Total differentiation of (3.5) with respect to t_1 and t_2, respectively, yields

$$f_x \frac{dx_1}{dt_1} + f_y \frac{dy_1}{dt_1} + f_z \frac{dz_1}{dt_1} = 0 \,, \tag{3.6}$$

$$f_x \frac{dx_2}{dt_2} + f_y \frac{dy_2}{dt_2} + f_z \frac{dz_2}{dt_2} = 0 \,. \tag{3.7}$$

Now if we multiply (3.6) by $\frac{dx_2}{dt_2}$ and subtract (3.7) multiplied by $\frac{dx_1}{dt_1}$, and if we multiply (3.6) by $\frac{dy_2}{dt_2}$ and subtract (3.7) multiplied by $\frac{dy_1}{dt_1}$ we can deduce the following relation

$$f_x : f_y : f_z = \tag{3.8}$$
$$\frac{dz_2}{dt_2}\frac{dy_1}{dt_1} - \frac{dz_1}{dt_1}\frac{dy_2}{dt_2} : \frac{dz_1}{dt_1}\frac{dx_2}{dt_2} - \frac{dz_2}{dt_2}\frac{dx_1}{dt_1} : \frac{dx_1}{dt_1}\frac{dy_2}{dt_2} - \frac{dx_2}{dt_2}\frac{dy_1}{dt_1} \,,$$

which indicates that vector $\nabla f = (f_x, f_y, f_z)^T$ (also known as gradient of f) is in the direction of the cross product of the two tangent vectors at P, i.e. in the normal direction. Thus the unit normal vector of the implicit surface is given by

$$\mathbf{N} = \frac{(f_x, f_y, f_z)^T}{\sqrt{f_x^2 + f_y^2 + f_z^2}} = \frac{\nabla f}{|\nabla f|} \,, \tag{3.9}$$

provided that $|\nabla f| \neq 0$.

Alternatively, we can derive (3.9) by considering an arbitrary parametric curve $\mathbf{r} = \mathbf{r}(t)$ on an implicit surface $f(x, y, z) = 0$, leading to the relation $\nabla f \cdot \dot{\mathbf{r}} = 0$. Since $\mathbf{r} = \mathbf{r}(t)$ is arbitrary, ∇f must be perpendicular to the tangent plane, and hence it is a normal vector.

The tangent plane of an implicit surface $f(x, y, z) = 0$ at point P with coordinates (x_p, y_p, z_p) can be obtained by replacing the normal vector of parametric surface in (3.4) with (3.9), which leads to

$$f_x(x - x_p) + f_y(y - y_p) + f_z(z - z_p) = 0 \,, \tag{3.10}$$

where $f(x_p, y_p, z_p) = 0$ and f_x, f_y f_z in (3.10) are evaluated at (x_p, y_p, z_p).

Example 3.1.2. The elliptic cone of Example 3.1.1 has also the following implicit representation $f(x, y, z) = (\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = 0$. The magnitude of the normal vector $\nabla f = (\frac{2x}{a^2}, \frac{2y}{b^2}, -\frac{2z}{c^2})^T$, where $(x, y, z) \in f(x, y, z) = 0$, becomes 0 only when x=y=z=0 corresponding to the apex of the cone as also derived in Example 3.1.1.

3.2 First fundamental form *I* (metric)

The differential arc length of a parametric curve is given by (2.2). Now if we replace the parametric curve by a curve $u = u(t)$, $v = v(t)$ which lies on the parametric surface $\mathbf{r} = \mathbf{r}(u, v)$, then

$$ds = \left|\frac{d\mathbf{r}}{dt}\right| dt = \left|\mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}\right| dt = \sqrt{(\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}) \cdot (\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v})}dt$$
$$= \sqrt{Edu^2 + 2Fdudv + Gdv^2}\ , \quad (3.11)$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v\ . \quad (3.12)$$

The *first fundamental form* is defined as

$$I = ds^2 = d\mathbf{r} \cdot d\mathbf{r} = Edu^2 + 2Fdudv + Gdv^2\ , \quad (3.13)$$

and E, F, G are called the first fundamental form coefficients and play important roles in many intrinsic properties of a surface. The first fundamental form I can be rewritten as

$$I = \frac{1}{E}(E\ du + F\ dv)^2 + \frac{EG - F^2}{E} dv^2\ . \quad (3.14)$$

Since $(\mathbf{r}_u \times \mathbf{r}_v)^2 = (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 = EG - F^2 > 0$[1] and $E = \mathbf{r}_u \cdot \mathbf{r}_u > 0$, I is positive definite, provided that the surface is regular. That is $I \geq 0$ and $I = 0$ if and only if $du = 0$ and $dv = 0$.

Example 3.2.1. Let us compute the arc length of a curve $u = t$, $v = t$ for $0 \leq t \leq 1$ on a hyperbolic paraboloid $\mathbf{r}(u, v) = (u, v, uv)^T$ where $0 \leq u, v \leq 1$ as shown in Fig. 3.4 (a). We have

$$\mathbf{r}_u = (1, 0, v)^T, \quad \mathbf{r}_v = (0, 1, u)^T\ ,$$
$$E = \mathbf{r}_u \cdot \mathbf{r}_u = 1 + v^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v = uv, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v = 1 + u^2\ ,$$

and along the curve the first fundamental form coefficients are

$$E = 1 + t^2, \quad F = t^2, \quad G = 1 + t^2\ ,$$

thus,

$$ds = \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}dt = 2\sqrt{t^2 + \frac{1}{2}}dt\ .$$

Finally the arc length for $0 \leq t \leq 1$ is given by

[1] Here the vector identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\ , \quad (3.15)$$

with the special case

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2\ , \quad (3.16)$$

is used.

$$s = 2\int_0^1 \sqrt{t^2 + \frac{1}{2}} dt = \left[t\sqrt{t^2 + \frac{1}{2}} + \frac{1}{2} log \left(t + \sqrt{t^2 + \frac{1}{2}} \right) \right]_0^1$$
$$= \sqrt{\frac{3}{2}} + \frac{1}{2} log(\sqrt{2} + \sqrt{3}) \ .$$

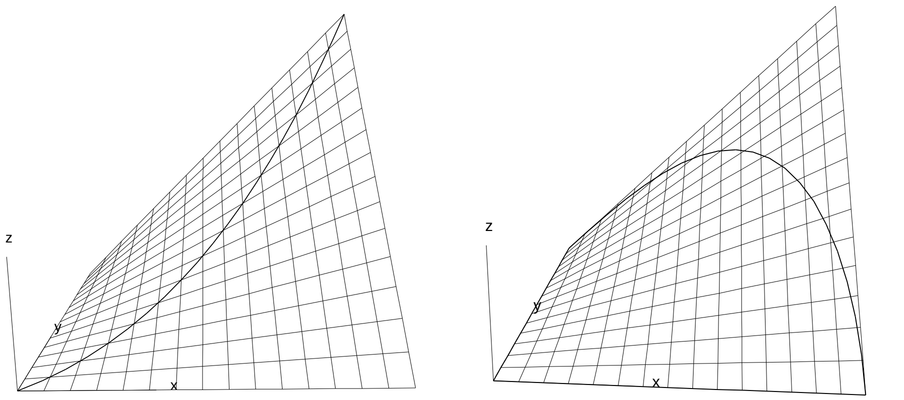

Fig. 3.4. Hyperbolic paraboloid: (**a**) arc length along $u = t$, $v = t$, (**b**) area bounded by positive u and v axes and a quarter circle

The angle between two curves on a parametric surface $\mathbf{r}_1 = \mathbf{r}(u_1(t), v_1(t))$ and $\mathbf{r}_2 = \mathbf{r}(u_2(t), v_2(t))$ can be evaluated by taking the inner product of the tangent vectors of $\mathbf{r}_1$ and $\mathbf{r}_2$, yielding

$$\cos\omega = \frac{Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2}{\sqrt{Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2}\sqrt{Edu_2^2 + 2Fdu_2dv_2 + Gdv_2^2}}$$
$$= E\frac{du_1}{ds_1}\frac{du_2}{ds_2} + F\left(\frac{du_1}{ds_1}\frac{dv_2}{ds_2} + \frac{dv_1}{ds_1}\frac{du_2}{ds_2}\right) + G\frac{dv_1}{ds_1}\frac{dv_2}{ds_2} \ . \quad (3.17)$$

As a result of the above equation, the orthogonality condition for the two tangent vectors $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ is:

$$Edu_1du_2 + F(du_1dv_2 + dv_1du_2) + Gdv_1dv_2 = 0 \ . \quad (3.18)$$

In particular when the two curves are the u and v iso-parametric curves, (3.17) reduces to

$$\cos\omega = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{|\mathbf{r}_u||\mathbf{r}_v|} = \frac{\mathbf{r}_u \cdot \mathbf{r}_v}{\sqrt{\mathbf{r}_u \cdot \mathbf{r}_u}\sqrt{\mathbf{r}_v \cdot \mathbf{r}_v}} = \frac{F}{\sqrt{EG}} \ . \quad (3.19)$$

Thus the iso-parametric curves are orthogonal if $F = 0$.

The area of a small parallelogram with vertices $\mathbf{r}(u, v)$, $\mathbf{r}(u+\delta u, v)$, $\mathbf{r}(u, v+\delta v)$ and $\mathbf{r}(u + \delta u, v + \delta v)$ as illustrated in Fig. 3.5, is approximated by

$$\delta A = |\mathbf{r}_u \delta u \times \mathbf{r}_v \delta v| = \sqrt{EG - F^2}\delta u \delta v \ , \tag{3.20}$$

or in differential form

$$dA = \sqrt{EG - F^2} du dv \ . \tag{3.21}$$

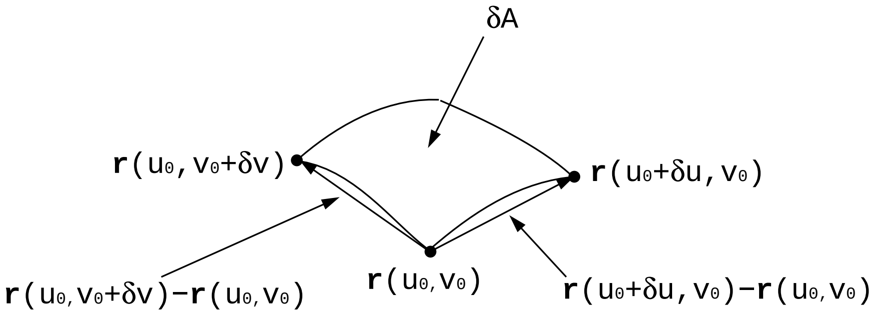

Fig. 3.5. Area of small surface patch

Example 3.2.2. Let us compute the area of a region of the hyperbolic paraboloid that is used in Example 3.2.1. The region is bounded by positive u and v axes and a quarter circle $u^2 + v^2 = 1$ as shown in Fig. 3.4 (b). Substituting $EG - F^2 = (1 + v^2)(1 + u^2) - u^2v^2 = 1 + u^2 + v^2$ into (3.21), we obtain

$$A = \int_D \sqrt{1 + u^2 + v^2} du dv \ .$$

To perform the integration it is easier to change variables, $u = r \cos\theta$, $v = r \sin\theta$, so that

$$A = \int_0^{\frac{\pi}{2}} \int_0^1 \sqrt{1 + r^2} \ r \ d\theta \ dr = \frac{\pi}{6}(\sqrt{8} - 1) \ .$$

3.3 Second fundamental form II (curvature)

In order to quantify the curvatures of a surface S, we consider a curve C on S which passes through point P as shown in Fig. 3.6. The unit tangent vector $\mathbf{t}$ and the unit normal vector $\mathbf{n}$ of the curve C at point P are related by (2.20) as follows:

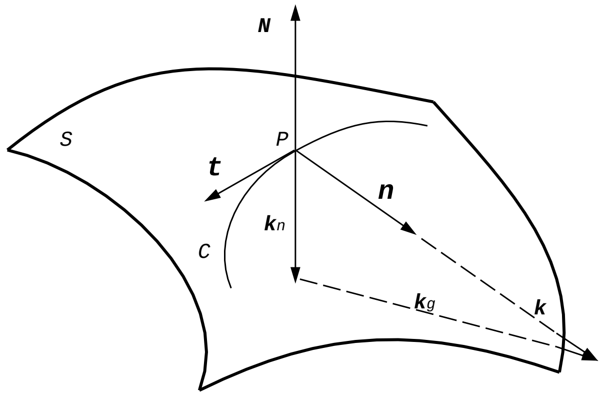

Fig. 3.6. Definition of normal curvature

$$\mathbf{k} = \frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} = \mathbf{k}_n + \mathbf{k}_g \ , \tag{3.22}$$

where $\mathbf{k}_n$ is the *normal curvature vector* and $\mathbf{k}_g$ is the *geodesic curvature vector* which are the components of the curvature vector $\mathbf{k}$ of C in the surface normal direction and in the direction perpendicular to $\mathbf{t}$ in the surface tangent plane. Thus, the normal curvature vector can be expressed as

$$\mathbf{k}_n = \kappa_n \mathbf{N} \ , \tag{3.23}$$

where κ_n is called the normal curvature of the surface at P in the direction $\mathbf{t}$. In other words, κ_n is the magnitude of the projection of $\mathbf{k}$ onto the surface normal at P, with a sign determined by the orientation of the surface normal at P.

By differentiating $\mathbf{N} \cdot \mathbf{t} = 0$ along the curve with respect to s we obtain

$$\frac{d\mathbf{t}}{ds} \cdot \mathbf{N} + \mathbf{t} \cdot \frac{d\mathbf{N}}{ds} = 0 \ , \tag{3.24}$$

thus

$$\kappa_n = \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} = -\mathbf{t} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r} \cdot d\mathbf{N}}{d\mathbf{r} \cdot d\mathbf{r}} \tag{3.25}$$

$$= \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2} \ , \tag{3.26}$$

where

$$L = -\mathbf{r}_u \cdot \mathbf{N}_u, \quad M = -\frac{1}{2}(\mathbf{r}_u \cdot \mathbf{N}_v + \mathbf{r}_v \cdot \mathbf{N}_u) = -\mathbf{r}_u \cdot \mathbf{N}_v = -\mathbf{r}_v \cdot \mathbf{N}_u \ ,$$
$$N = -\mathbf{r}_v \cdot \mathbf{N}_v \ . \tag{3.27}$$

Since $\mathbf{r}_u$ and $\mathbf{r}_v$ are perpendicular to $\mathbf{N}$, we have $\mathbf{r}_u \cdot \mathbf{N} = 0$ and $\mathbf{r}_v \cdot \mathbf{N} = 0$, and hence we have an alternative expression for L, M and N

$$L = \mathbf{r}_{uu} \cdot \mathbf{N}, \quad M = \mathbf{r}_{uv} \cdot \mathbf{N}, \quad N = \mathbf{r}_{vv} \cdot \mathbf{N} . \tag{3.28}$$

Computation of curvatures at points where the surface representation is degenerate (see Sect. 1.3.6) is given in [453].

The numerator of (3.26) is the *second fundamental form II*, i.e.

$$II = Ldu^2 + 2Mdudv + Ndv^2 , \tag{3.29}$$

and L, M, N are called second fundamental form coefficients. Therefore the normal curvature is given by

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} , \tag{3.30}$$

where $\lambda = \frac{dv}{du}$ is the direction of the tangent line to C at P. We can observe that κ_n at a given point P on the surface depends only on λ which leads to the following theorem due to Meusnier.

Theorem 3.3.1. *All curves lying on a surface S passing through a given point $p \in S$ with the same tangent line have the same normal curvature at this point.*

Using this theorem we can say that the normal curvature is positive when the center of the curvature of the normal section curve, which is a curve through P cut out by a plane that contains $\mathbf{t}$ and $\mathbf{N}$, is on the same side of the surface normal (see Fig. 3.7 (a)). Sometimes the positive normal curvature is defined in the opposite direction, i.e. the center of curvature of the normal section curve is on the opposite side of the surface normal as illustrated in Fig. 3.7 (b). In such cases (3.23) (3.30) become

$$\mathbf{k}_n = -\kappa_n \mathbf{N}, \quad \kappa_n = -\frac{II}{I} = -\frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} . \tag{3.31}$$

The latter convention is often used in the area of offset curves and surfaces in the context of NC machining. Throughout this book we refer to the first convention as convention (a) and to the second one as convention (b). We have listed all the equations, which involve changes due to this convention in the last page of this chapter.

Suppose P is a point on a surface and Q is a point in the neighborhood of P and $\mathbf{r} = \mathbf{r}(u, v)$ is the surface containing P and Q, as in Fig. 3.8. Now suppose P and Q are the points $\mathbf{r}(u, v)$ and $\mathbf{r}(u + du, v + dv)$, then Taylor's expansion gives

$$\begin{aligned} \mathbf{r}(u + du, v + dv) = \; & \mathbf{r}(u, v) + \mathbf{r}_u du + \mathbf{r}_v dv \\ & + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} dudv + \mathbf{r}_{vv} dv^2) + \ldots . \end{aligned} \tag{3.32}$$

Therefore

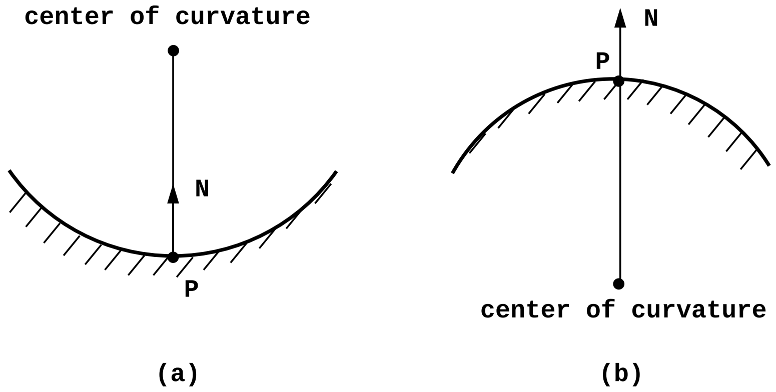

Fig. 3.7. Definition of positive normal curvature: (**a**) $\kappa\mathbf{n}\cdot\mathbf{N} = \kappa_n$, (**b**) $\kappa\mathbf{n}\cdot\mathbf{N} = -\kappa_n$

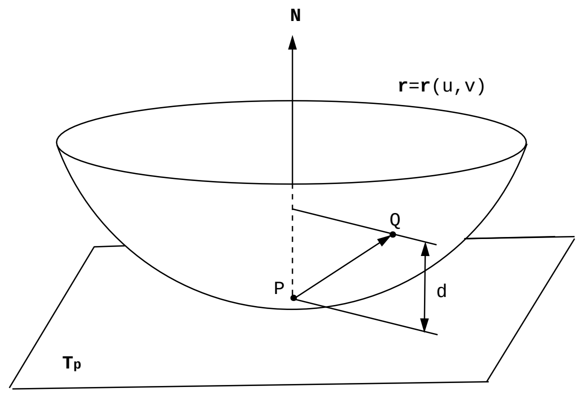

Fig. 3.8. Geometrical illustration of the second fundamental form

$$\begin{aligned}\mathbf{PQ} = \ & \mathbf{r}(u+du, v+dv) - \mathbf{r}(u,v) = \mathbf{r}_u du + \mathbf{r}_v dv \\ & +\frac{1}{2}(\mathbf{r}_{uu}du^2 + 2\mathbf{r}_{uv}dudv + \mathbf{r}_{vv}dv^2) + \dots \ .\end{aligned} \tag{3.33}$$

Thus using (3.28), (3.29), the projection of $\mathbf{PQ}$ onto $\mathbf{N}$ is

$$d = \mathbf{PQ}\cdot\mathbf{N} = (\mathbf{r}_u du + \mathbf{r}_v dv)\cdot\mathbf{N} + \frac{1}{2}II \ , \tag{3.34}$$

where the higher order terms are neglected and since $\mathbf{r}_u \cdot \mathbf{N} = \mathbf{r}_v \cdot \mathbf{N} = 0$, we get

$$d = \frac{1}{2}II = \frac{1}{2}(Ldu^2 + 2Mdudv + Ndv^2) \ . \tag{3.35}$$

Thus $|II|$ is equal to twice the distance from Q to the tangent plane of the surface at P within second order terms. We want to observe in which situation d is positive and negative or in other words we want to examine in which side of the tangent plane Q lies. When $d = 0$, (3.35) becomes $Ldu^2 + 2Mdudv + Ndv^2 = 0$, which can be considered as a quadratic equation in terms of du or dv. If we solve for du, assuming $L \neq 0$, we obtain

$$du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv \,, \tag{3.36}$$

which leads us to the following four cases:

- If $M^2 - LN < 0$, there is no real root. This means there is no intersection between the surface and its tangent plane except at point P. Point P is called *elliptic point* (Fig. 3.9(a)). For example, an ellipsoid consists entirely of elliptic points.
- If $M^2 - LN = 0$ and $L^2 + M^2 + N^2 \neq 0$, there are double roots. The surface intersects its tangent plane with one line $du = -\frac{M}{L}dv$, which passes through point P. Point P is called *parabolic point* (Fig. 3.9(b)). For example, a circular cylinder consists entirely of parabolic points.
- If $M^2 - LN > 0$, there are two roots. The surface intersects its tangent plane with two lines $du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv$, which intersect at point P. Point P is called *hyperbolic point* (Fig. 3.9(c)). For example, a hyperboloid of revolution consists entirely of hyperbolic points.
- If $L = M = N = 0$, the surface and the tangent plane have a contact of higher order than in the preceding cases. Point P is called a *flat* or *planar point.*

If $L = 0$ and $N \neq 0$, we can solve for dv instead of du. If $L = N = 0$ and $M \neq 0$, we have $2Mdudv = 0$, thus the iso-parametric lines $u = constant$, $v = constant$ will be the two intersection lines.

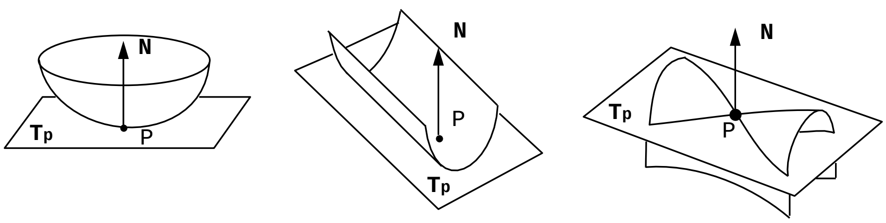

Fig. 3.9. (**a**) Elliptic point; (**b**) parabolic point; (**c**) hyperbolic point

3.4 Principal curvatures

As we can see from (3.30) the normal curvature at a point P depends on the direction of $\lambda = \frac{dv}{du}$. Now we will seek the directions in which the extrema of principal curvature occur following Struik [412]. The extreme values of κ_n can be obtained by evaluating $\frac{d\kappa_n}{d\lambda} = 0$ of (3.30), which gives:

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0 \; , \tag{3.37}$$

and hence

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} \; . \tag{3.38}$$

Furthermore since

$$\begin{aligned} E + 2F\lambda + G\lambda^2 &= (E + F\lambda) + \lambda(F + G\lambda) \; , \\ L + 2M\lambda + N\lambda^2 &= (L + M\lambda) + \lambda(M + N\lambda) \; , \end{aligned}$$

(3.37) can be reduced to

$$(E + F\lambda)(M + N\lambda) = (L + M\lambda)(F + G\lambda) \; , \tag{3.39}$$

and hence

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda} \; . \tag{3.40}$$

Therefore, the extreme values of κ_n satisfy the two simultaneous equations

$$\begin{aligned} &(L - \kappa_n E)du + (M - \kappa_n F)dv = 0 \; , \\ &(M - \kappa_n F)du + (N - \kappa_n G)dv = 0 \; . \end{aligned} \tag{3.41}$$

These equations form a homogeneous linear system of equations for du, dv, which will have a nontrivial solution if and only if

$$\begin{vmatrix} L - \kappa_n E & M - \kappa_n F \\ M - \kappa_n F & N - \kappa_n G \end{vmatrix} = 0 \; , \tag{3.42}$$

where $| \;\; |$ denotes the determinant of a matrix, or expanding

$$(EG - F^2)\kappa_n^2 - (EN + GL - 2FM)\kappa_n + (LN - M^2) = 0 \; . \tag{3.43}$$

The discriminant D of this quadratic equation in κ_n can be re-formulated as

$$D = 4\left(\frac{EG - F^2}{E^2}\right)(EM - FL)^2 + \left(EN - GL - \frac{2F}{E}(EM - FL)\right)^2 \; , \tag{3.44}$$

after some algebraic manipulations. Thus the discriminant D is always greater than or equal to zero and (3.43) has real roots. The discriminant D becomes zero if and only if $EM - FL = 0$ and $EN - GL = 0$ or if and only if there is a constant k such that

$$L = kE, \quad M = kF, \quad N = kG \; . \tag{3.45}$$

Such a point is called an *umbilic* and the normal curvature is the same in all directions. Therefore (3.43) has either two distinct real roots, or a double root. If we set

$$K = \frac{LN - M^2}{EG - F^2} \; , \tag{3.46}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} \; , \tag{3.47}$$

the quadratic equation for κ_n (3.43) simplifies to:

$$\kappa_n^2 - 2H\kappa_n + K = 0 \; . \tag{3.48}$$

The quantities K and H are called *Gaussian (Gauss) curvature* and *mean curvature*, respectively. Upon solving (3.48) for the extreme values of curvature, we have

$$\kappa_{max} = H + \sqrt{H^2 - K} \; , \tag{3.49}$$

$$\kappa_{min} = H - \sqrt{H^2 - K} \; , \tag{3.50}$$

where κ_{max} is the *maximum principal curvature* and κ_{min} is the *minimum principal curvature*. The directions in the tangent plane for which κ_n takes maximum and minimum values are called *principal directions*. The corresponding directions in the uv-plane can be determined by using (3.40), which leads to

$$\lambda = -\frac{M - \kappa_n F}{N - \kappa_n G} \; , \tag{3.51}$$

or

$$\lambda = -\frac{L - \kappa_n E}{M - \kappa_n F} \; , \tag{3.52}$$

where κ_n is replaced by either κ_{max} or κ_{min}.

When the discriminant is zero or $H^2 = K$, κ_n is a double root with value equal to H and the corresponding point of the surface is an *umbilical point*. At an umbilical point a surface is locally a part of sphere with radius of curvature $\frac{1}{|H|}$. In the special case where both K and H vanish, the point is a *flat* or *planar point*.

Alternatively we can derive the principal directions by solving a quadratic equation in λ

$$(FN - GM)\lambda^2 + (EN - GL)\lambda + (EM - FL) = 0 \ , \tag{3.53}$$

which is deduced from (3.37). The discriminant of this equation is easily shown to be the same as that of (3.43), and hence it is greater than or equal to zero. At an umbilical point the discriminant vanishes and (3.45) hold, thus we have $FN = GM$, $EN = GL$ and $EM = FL$. Therefore, the coefficients of the quadratic equation become all zero and thus the principal directions are not defined. When a point P on the surface is a non-umbilical point, there are always two principal directions determined by the quadratic equations. Let λ_{max} and λ_{min} be the directions of maximum and minimum principal curvature in the uv-plane. Then, λ_{max} and λ_{min} satisfy the quadratic equation (3.53):

$$(FN - GM)\lambda_{max}^2 + (EN - GL)\lambda_{max} + (EM - FL) = 0 \ , \tag{3.54}$$

$$(FN - GM)\lambda_{min}^2 + (EN - GL)\lambda_{min} + (EM - FL) = 0 \ . \tag{3.55}$$

From these equations we can deduce

$$\lambda_{max} + \lambda_{min} = -\frac{EN - GL}{FN - GM} \ , \tag{3.56}$$

$$\lambda_{max}\lambda_{min} = \frac{EM - FL}{FN - GM} \ , \tag{3.57}$$

thus,

$$\begin{aligned} & E + F(\lambda_{max} + \lambda_{min}) + G\lambda_{max}\lambda_{min} \\ & = \frac{1}{FN - GM}[E(FN - GM) - F(EN - GL) + G(EM - FL)] = 0 \ . \end{aligned} \tag{3.58}$$

Consequently, it is evident from (3.18) that the two tangent vectors in the principal directions are orthogonal.

A curve on a surface whose tangent at each point is in a principal direction at that point is called a *line of curvature*. Since at each (non-umbilical) point there are two principal directions that are orthogonal, the lines of curvatures form an orthogonal net of lines. Figure 3.10 shows an example of the lines of curvature on a saddle-shaped surface where all points are hyperbolic. The solid lines correspond to the maximum principal curvature direction, while the dashed lines correspond to the minimum principal curvature direction (convention (a) is used). Since there is no umbilical point on the surface, we do not encounter any singularity on the net of lines of curvature. The lines of curvature in the presence of umbilical points are discussed in Chap. 9.

This orthogonal net of lines can be used as a parametrization of a surface. In such cases, we have $F = 0$ (see (3.19)), and (3.41) reduce to

$$(L - \kappa_n E)du + Mdv = 0, \quad Mdu + (N - \kappa_n G)dv = 0 \ . \tag{3.59}$$

If these equations are satisfied by $du = 0$ and by $dv = 0$, this implies $M = 0$, and the two principal curvatures are $\kappa_1 = \frac{L}{E}$ and $\kappa_2 = \frac{N}{G}$, in the absence of

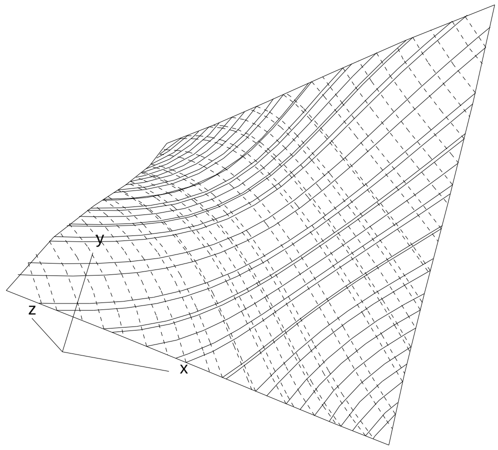

Fig. 3.10. Lines of curvature

umbilical points. Therefore the necessary condition for the parametric lines to be lines of curvature is

$$F = M = 0\ . \tag{3.60}$$

The converse is also true and the condition is also sufficient.

Example 3.4.1. As a curve C in the xz-plane $x = f(t)$, $z = g(t)$ revolves about the z-axis, it generates a *surface of revolution* S. The curves C in different rotated positions are called the *meridians* of S, while the circles generated by each point on C are called the *parallels* of S. If we denote the rotation angle in the xy-plane as θ, the surface of revolution can be parametrized as

$$\mathbf{r} = (f(t)\cos\theta, f(t)\sin\theta, g(t))^T\ .$$

Thus,

$$\mathbf{r}_t = (\dot{f}(t)\cos\theta, \dot{f}(t)\sin\theta, \dot{g}(t))^T, \qquad \mathbf{r}_\theta = (-f(t)\sin\theta, f(t)\cos\theta, 0)^T,$$

and hence

$$E = \dot{f}^2(t) + \dot{g}^2(t), \qquad F = 0, \qquad G = f^2(t)\ .$$

Since $F = 0$, (3.19) shows that the meridians and parallels are orthogonal. Furthermore we have

$$L = \frac{-\ddot{f}\dot{g} + \dot{f}\ddot{g}}{\sqrt{\dot{f}^2(t) + \dot{g}^2(t)}}, \qquad M = 0, \qquad N = \frac{f\dot{g}}{\sqrt{\dot{f}^2(t) + \dot{g}^2(t)}}\ ,$$

which lead us to the conclusion that the meridians and parallels of a surface of revolution are the lines of curvature.

3.5 Gaussian and mean curvatures

From (3.49), (3.50), it is readily seen that the Gaussian and mean curvatures are the product and the average of the two principal curvatures, respectively:

$$K = \kappa_{max}\kappa_{min} \ , \tag{3.61}$$

$$H = \frac{\kappa_{max} + \kappa_{min}}{2} \ . \tag{3.62}$$

The sign of the Gaussian curvature coincides with sign of $LN - M^2$, since $K = \frac{LN-M^2}{EG-F^2}$ (see (3.46)) and $EG - F^2 > 0$. Consequently a point on a surface is elliptic if $K > 0$ (κ_{max} and κ_{min} are of the same sign), hyperbolic if $K < 0$ (κ_{max} and κ_{min} have different signs) and parabolic if $K = 0$ and $H \neq 0$ (either κ_{max} or κ_{min} is zero), flat or planar point if $K = H = 0$ ($\kappa_{max} = \kappa_{min} = 0$).

3.5.1 Explicit surfaces

Very often a surface is given by an explicit form $z = h(x, y)$. It is, therefore, convenient to have analytic equations for the Gaussian and mean curvatures expressed in terms of the derivatives of the height function $h(x, y)$. As we mentioned in Sect. 1.1 the explicit form can be converted into a parametric form $\mathbf{r} = (u, v, h(u, v))^T$ where $u = x$ and $v = y$. This form is often referred to as *Monge form* , and the surface is called a Monge patch. It is straightforward to evaluate

$$E = 1 + h_x^2, \quad F = h_x h_y, \quad G = 1 + h_y^2 \ , \tag{3.63}$$

$$\mathbf{N} = \frac{(-h_x, -h_y, 1)^T}{\sqrt{1 + h_x^2 + h_y^2}} \ , \tag{3.64}$$

$$L = \frac{h_{xx}}{\sqrt{1 + h_x^2 + h_y^2}}, \quad M = \frac{h_{xy}}{\sqrt{1 + h_x^2 + h_y^2}}, \quad N = \frac{h_{yy}}{\sqrt{1 + h_x^2 + h_y^2}} \ , \tag{3.65}$$

and hence

$$K = \frac{LN - M^2}{EG - F^2} = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2} \ , \tag{3.66}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{(1 + h_x^2)h_{yy} - 2h_x h_y h_{xy} + (1 + h_y^2)h_{xx}}{2(1 + h_x^2 + h_y^2)^{3/2}} \ . \tag{3.67}$$

Example 3.5.1. Let us compute the Gaussian and mean curvatures of the hyperbolic paraboloid $z = xy$ (in Example 3.2.1 we used its parametric form) using the explicit formulae (3.63) to (3.67). Since

$$h_x = y, \quad h_y = x, \quad h_{xx} = 0, \quad h_{xy} = 1, \quad h_{yy} = 0, \quad \mathbf{N} = \frac{(-y,-x,1)^T}{\sqrt{x^2+y^2+1}} \,,$$

we have

$$E = 1+y^2, \quad F = xy, \quad G = 1+x^2, \quad L = 0, \quad M = \frac{1}{\sqrt{x^2+y^2+1}}, \quad N = 0 \,,$$

and hence

$$K = -\frac{1}{(x^2+y^2+1)^2}, \qquad H = -\frac{xy}{(x^2+y^2+1)^{\frac{3}{2}}} \,.$$

Here we can observe that the Gaussian curvature is always negative and thus all the points on a hyperbolic paraboloid are hyperbolic points. Furthermore, since $L = N = 0$ and $M \neq 0$, the surface intersects its tangent plane at the iso-parametric lines (see Sect. 3.3 last paragraph). Also from (3.49) and (3.50) we obtain

$$\kappa_{max} = \frac{-xy + \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{\frac{3}{2}}}, \qquad \kappa_{min} = \frac{-xy - \sqrt{(x^2+1)(y^2+1)}}{(x^2+y^2+1)^{\frac{3}{2}}} \,,$$

where it is very easy to show that $\kappa_{max} > 0$ and $\kappa_{min} < 0$ for all (x, y).

3.5.2 Implicit surfaces

Using (3.66), (3.67) for an explicit surface, we can derive equations for the Gaussian and mean curvature of an implicit surface $f(x,y,z) = 0$. At a point where $f_z \neq 0$, z can be expressed as a function of x and y, say $z = h(x,y)$ [166]. In such cases variables x and y are independent but z is a function of both x and y. Since f constantly satisfies the equation $f(x,y,z) = 0$, the partial differentiation of f with respect to the independent variable x (by holding y fixed) must vanish [166]. Thus,

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \,, \tag{3.68}$$

where $\left(\frac{\partial f}{\partial x}\right)_y$ on the left-hand side is considered as f being expressed in terms of x and y only and y is held constant in the differentiation with respect to x, while $\frac{\partial f}{\partial x}$ on the right-hand side is considered as f being expressed in terms of x, y, z and y, z are held constant in the x differentiation. Similarly we have

$$\left(\frac{\partial f}{\partial y}\right)_x = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \ . \tag{3.69}$$

Consequently we have

$$h_x = -\frac{f_x}{f_z}, \qquad h_y = -\frac{f_y}{f_z} \ . \tag{3.70}$$

The second order partial derivatives h_{xx}, h_{xy}, h_{yy} are provided by differentiating (3.70). For example,

$$h_{xx} = -\frac{\left(\frac{\partial f_x}{\partial x}\right)_y f_z - \left(\frac{\partial f_z}{\partial x}\right)_y f_x}{f_z^2} = \frac{2f_xf_zf_{xz} - f_x^2f_{zz} - f_z^2f_{xx}}{f_z^3} \ . \tag{3.71}$$

Similarly, we have

$$h_{xy} = \frac{f_xf_zf_{yz} + f_yf_zf_{xz} - f_xf_yf_{zz} - f_z^2f_{xy}}{f_z^3} \ , \tag{3.72}$$

$$h_{yy} = \frac{2f_yf_zf_{yz} - f_y^2f_{zz} - f_z^2f_{yy}}{f_z^3} \ . \tag{3.73}$$

Equations (3.70) to (3.73) may be substituted into (3.63) and (3.65) to obtain the first and second fundamental form coefficients, and into (3.66) and (3.67) to compute the Gaussian and mean curvature of an implicit surface. If $f_z = 0$, alternate formulae may be found by cyclic permutation of x, y, z.

For every quadric surface, it is possible to find a suitable 3-D rotation such that the cross terms dxy, eyz and fxz cancel out in (1.15). If a quadric surface has a center[2], its axes can be translated to the center as origin so that the equation of the quadric surface does not have any first degree terms [79]. Therefore after these transformations the implicit quadrics, ellipsoids, hyperboloids of one and two sheets, elliptic cones, elliptic cylinders and hyperbolic cylinders can be expressed in a *standard form*

$$f(x, y, z) = \zeta\frac{x^2}{a^2} + \eta\frac{y^2}{b^2} + \xi\frac{z^2}{c^2} - \delta = 0 \ , \tag{3.74}$$

where ζ, η and ξ take values either -1, 0 or 1 and δ takes values either 0 or 1, depending on the classification of quadrics (see Table 3.1).

By evaluating (3.70), (3.71), (3.72) and (3.73) for f given in (3.74), and substituting into (3.66) and (3.67), we obtain

[2] A center of a quadric surface is defined as a point bisecting every chord passing through it [79]. Here chord is a line which joins two points on a surface. Ellipsoids and hyperboloids have centers, while paraboloids do not have centers. The elliptic/hyperbolic cylinder is a limiting case of the ellipsoid/hyperboloid and the elliptic cone is asymptotic to hyperboloids of one and two sheets.

Table 3.1. Classification of implicit quadrics

Implicit Quadrics	ζ	η	ξ	δ
Ellipsoid	1	1	1	1
Hyperboloid of One Sheet	1	1	-1	1
	1	-1	1	1
	-1	1	1	1
Hyperboloid of Two Sheets	1	-1	-1	1
	-1	1	-1	1
	-1	-1	1	1
Elliptic Cone	1	1	-1	0
	1	-1	1	0
	-1	1	1	0
Elliptic Cylinder	1	1	0	1
	1	0	1	1
	0	1	1	1
Hyperbolic Cylinder	1	-1	0	1
	-1	1	0	1
	1	0	-1	1
	-1	0	1	1
	0	1	-1	1
	0	-1	1	1

$$K(x,y,z) = \frac{\zeta\eta\xi\delta}{a^2b^2c^2(\zeta^2\frac{x^2}{a^4}+\eta^2\frac{y^2}{b^4}+\xi^2\frac{z^2}{c^4})^2}\ , \tag{3.75}$$

$$H(x,y,z) = \tag{3.76}$$
$$-\frac{\zeta^2b^2c^2(\xi b^2+\eta c^2)x^2+\eta^2a^2c^2(\xi a^2+\zeta c^2)y^2+\xi^2a^2b^2(\eta a^2+\zeta b^2)z^2}{2a^4b^4c^4(\zeta^2\frac{x^2}{a^4}+\eta^2\frac{y^2}{b^4}+\xi^2\frac{z^2}{c^4})^{\frac{3}{2}}}\ ,$$

where (x,y,z) satisfy $f(x,y,z)=0$. The principal curvatures can be obtained by substituting (3.75) and (3.77) into

$$\kappa(x,y,z) = H \pm \sqrt{H^2-K}\ , \tag{3.77}$$

where we will not show the substituted expression because it is too cumbersome.

The curvatures of a hyperbolic cylinder ($\zeta=\delta=1$, $\eta=-1$, $\xi=0$)

$$f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0\ , \tag{3.78}$$

can be obtained by evaluating (3.75), (3.77) and (3.77) resulting

$$K=0, \qquad H = \frac{b^2x^2-a^2y^2}{2a^4b^4(\frac{x^2}{a^4}+\frac{y^2}{b^4})^{\frac{3}{2}}}\ , \tag{3.79}$$

$$\kappa_{max} = \frac{b^2x^2-a^2y^2}{a^4b^4(\frac{x^2}{a^4}+\frac{y^2}{b^4})^{\frac{3}{2}}}, \qquad \kappa_{min}=0\ , \tag{3.80}$$

where $(x,y) \in f(x,y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$.

Similarly, the curvatures of an ellipsoid ($\zeta = \eta = \xi = \delta = 1$)

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0\,, \tag{3.81}$$

are evaluated as

$$K = \frac{1}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^2}, \quad H = \frac{x^2+y^2+z^2-a^2-b^2-c^2}{2a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}\,, \tag{3.82}$$

$$\kappa = \frac{x^2+y^2+z^2-a^2-b^2-c^2}{2a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}} \pm \frac{\sqrt{(x^2+y^2+z^2-a^2-b^2-c^2)^2 - 4a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)}}{2a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}\,, \tag{3.83}$$

where $(x,y,z) \in f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$. Here we note that in the derivation of the mean curvature in (3.82), we used (3.81) to simplify the expression. For the case of a sphere of radius R, (3.81) simplifies to $f(x,y,z) = \frac{1}{R^2}(x^2+y^2+z^2) - 1 = 0$, and (3.82) and (3.83) simplify to $K = \frac{1}{R^2}$, $H = \kappa = -\frac{1}{R}$, which shows that a sphere is made of entirely nonflat umbilics (see Sects. 9.1 and 9.2). The negative sign comes from the sign convention of the curvature (see Fig. 3.7 and Table 3.2).

Finally, the curvatures of an elliptic cone ($\zeta = \eta = 1$, $\xi = -1$ and $\delta = 0$)

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0\,, \tag{3.84}$$

excluding the apex (0,0,0) are given by

$$K = 0, \quad H = -\frac{x^2+y^2+z^2}{2a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}\,, \tag{3.85}$$

$$\kappa_{max} = 0, \kappa_{min} = -\frac{x^2+y^2+z^2}{a^2b^2c^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{3}{2}}}, \tag{3.86}$$

where $(x,y,z) \in f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$. Here we also used (3.84) to simplify the expression of mean curvature in (3.85).

3.6 Euler's theorem and Dupin's indicatrix

The normal curvatures of a surface in an arbitrary direction (in the tangent plane) at point P can be expressed in terms of principal curvatures κ_1 and κ_2

at point P and the angle Φ between the arbitrary direction and the principal direction corresponding to κ_1, namely,

$$\kappa_n = \kappa_1 \cos^2 \Phi + \kappa_2 \sin^2 \Phi \ . \tag{3.87}$$

This is known as *Euler's theorem.* For simplicity, we assume that the iso-parametric curves of a surface are lines of curvature, which leads to $F = M = 0$ (see (3.60)). Now (3.26) takes the form

$$\kappa_n = \frac{Ldu^2 + Ndv^2}{Edu^2 + Gdv^2} \ . \tag{3.88}$$

For $v = const$ iso-parametric lines $dv = 0$ and for $u = const$ iso-parametric lines $du = 0$, thus the principal curvatures κ_1 and κ_2 are given by:

$$\kappa_1 = \frac{L}{E}, \qquad \kappa_2 = \frac{N}{G} \ . \tag{3.89}$$

The angle Φ between the direction $\frac{dv}{du}$ and the principal direction corresponding to κ_1 ($dv_1 = 0$, u_1 arbitrary) is evaluated by (3.17) as

$$\cos\Phi = E\frac{du}{ds}\frac{du_1}{ds_1} \ . \tag{3.90}$$

Since $ds_1 = \sqrt{Edu_1^2}$ and $ds = \sqrt{Edu^2 + Gdv^2}$ we deduce

$$\cos\Phi = \sqrt{E}\frac{du}{ds}, \qquad \sin\Phi = \sqrt{G}\frac{dv}{ds} \ . \tag{3.91}$$

As a consequence, we have the Euler's theorem (3.87).

Next we explain Euler's theorem in a more simple way. Let us consider a section of the surface cut by a plane parallel to the tangent plane at the point P, and at an infinitesimal distance $h > 0$ from it [441]. We also consider a plane through P containing the normal vector. If we denote the intersection points of the surface and the two planes by Q and Q', the signed radius of curvature of this normal section by ϱ, and the length of QQ' by $2R$ as shown in Fig. 3.11, we have the relation

$$(|\varrho| - h)^2 + R^2 = |\varrho|^2 \ , \tag{3.92}$$

thus

$$R^2 = 2h|\varrho| \ , \tag{3.93}$$

to the first order. If Φ is the inclination of this normal section to the principal direction corresponding to κ_1, Euler's theorem provides

$$\kappa_1 \cos^2 \Phi + \kappa_2 \sin^2 \Phi = \frac{1}{\varrho} = \pm\frac{2h}{R^2} \ . \tag{3.94}$$

If we set

$$\xi = R\cos\Phi, \qquad \eta = R\sin\Phi\ , \tag{3.95}$$

we obtain

$$\frac{\xi^2}{2h\varrho_1} + \frac{\eta^2}{2h\varrho_2} = \pm 1\ , \tag{3.96}$$

where ϱ_1 and ϱ_2 are principal radius of curvatures. Consequently a section of the surface cut by a plane parallel to the tangent plane at the point P, and at an infinitesimal distance is a conic section. If we scale the ξ-η coordinates as follows

$$X = \frac{\xi}{\sqrt{2h}} = \frac{R}{\sqrt{2h}}\cos\Phi = \sqrt{|\varrho|}\cos\Phi, \tag{3.97}$$

$$Y = \frac{\eta}{\sqrt{2h}} = \frac{R}{\sqrt{2h}}\sin\Phi = \sqrt{|\varrho|}\sin\Phi\ , \tag{3.98}$$

we obtain

$$\frac{X^2}{\varrho_1} + \frac{Y^2}{\varrho_2} = \pm 1\ . \tag{3.99}$$

This equation determines a conic section called *Dupin's indicatrix* as shown in Fig. 3.12. If P is an elliptic point, both principal curvatures have the same sign, and the indicatrix is an ellipse, while if it is a hyperbolic point, the principal curvatures have different sign and the indicatrix consists of a pair of hyperbolas with asymptotic lines $Y = \pm\sqrt{\frac{|\varrho_2|}{|\varrho_1|}}X$. If one of the principal curvatures vanishes, it is a parabolic point and the indicatrix yields a pair of parallel lines.

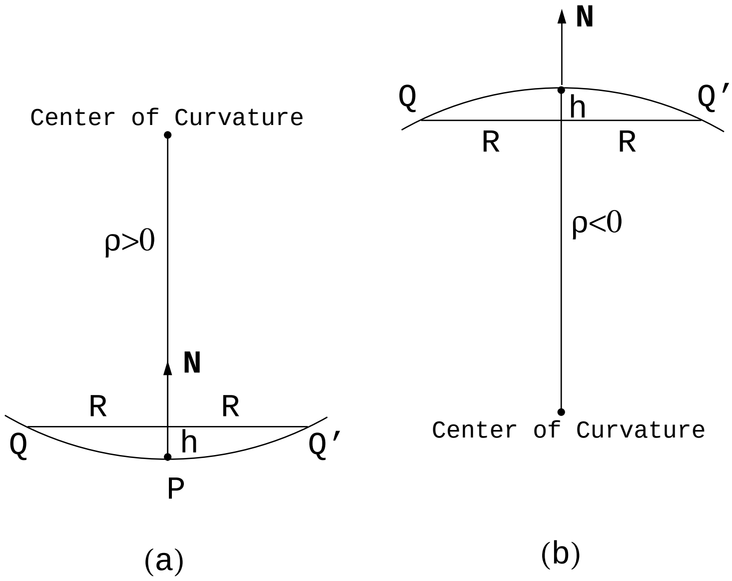

Fig. 3.11. Cross section of the surface cut by a normal plane: (**a**) normal curvature is positive, (**b**) normal curvature is negative (Here we followed the curvature convention (a); see Fig. 3.7)

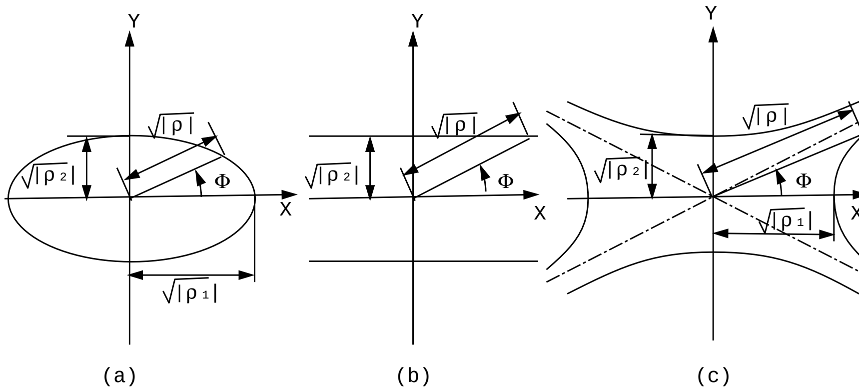

Fig. 3.12. Dupin's indicatrix for (**a**) elliptic point, (**b**) parabolic point, (**c**) hyperbolic point

Table 3.2. A list of equations which involves a sign change due to the sign convention of curvature of the planar curve or the normal curvature of the surface (see Fig. 3.7). In sign convention (a) the center of curvature is on the same side of the normal vector, while in sign convention (b) it is on the opposite direction

Equation	Convention (a)	Convention (b)
(2.20)	$\mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}$	$\mathbf{r}'' = \mathbf{t}' = -\kappa\mathbf{n}$
(2.22)	$\ddot{\mathbf{r}} = \kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt}$	$\ddot{\mathbf{r}} = -\kappa\mathbf{n}v^2 + \mathbf{t}\frac{dv}{dt}$
(2.24)	$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(-\dot{y},\dot{x})^T}{\sqrt{\dot{x}^2+\dot{y}^2}}$	$\mathbf{n} = \mathbf{t} \times \mathbf{e}_z = \frac{(\dot{y},-\dot{x})^T}{\sqrt{\dot{x}^2+\dot{y}^2}}$
(2.27)	$\mathbf{n} = \mathbf{e}_z \times \mathbf{t} = \frac{(f_x,f_y)^T}{\sqrt{f_x^2+f_y^2}}$ $= \frac{\nabla f}{\lvert\nabla f\rvert}$	$\mathbf{n} = \mathbf{t} \times \mathbf{e}_z = \frac{(-f_x,-f_y)^T}{\sqrt{f_x^2+f_y^2}}$ $= -\frac{\nabla f}{\lvert\nabla f\rvert}$
(2.55)	$\mathbf{n}' = -\kappa\mathbf{t}\ (\tau = 0)$	$\mathbf{n}' = \kappa\mathbf{t}\ (\tau = 0)$
(3.23)	$\mathbf{k}_n = \kappa_n\mathbf{N}$	$\mathbf{k}_n = -\kappa_n\mathbf{N}$
(3.25)	$\kappa_n = \frac{d\mathbf{t}}{ds}\cdot\mathbf{N} = -\mathbf{t}\cdot\frac{d\mathbf{N}}{ds}$ $= -\frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r}\cdot d\mathbf{N}}{d\mathbf{r}\cdot d\mathbf{r}}$	$\kappa_n = -\frac{d\mathbf{t}}{ds}\cdot\mathbf{N} = \mathbf{t}\cdot\frac{d\mathbf{N}}{ds}$ $= \frac{d\mathbf{r}}{ds}\cdot\frac{d\mathbf{N}}{ds} = \frac{d\mathbf{r}\cdot d\mathbf{N}}{d\mathbf{r}\cdot d\mathbf{r}}$
(3.26)	$\kappa_n = \frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$	$\kappa_n = -\frac{Ldu^2+2Mdudv+Ndv^2}{Edu^2+2Fdudv+Gdv^2}$
(3.30)	$\kappa_n = \frac{II}{I} = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$	$\kappa_n = -\frac{II}{I} = -\frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$
(3.38)	$\kappa_n = \frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$ $= \frac{M+N\lambda}{F+G\lambda}$	$\kappa_n = -\frac{L+2M\lambda+N\lambda^2}{E+2F\lambda+G\lambda^2}$ $= -\frac{M+N\lambda}{F+G\lambda}$
(3.41)	$(L-\kappa_n E)du + (M-\kappa_n F)dv$ $= 0$ $(M-\kappa_n F)du + (N-\kappa_n G)dv$ $= 0$	$(L+\kappa_n E)du + (M+\kappa_n F)dv$ $= 0$ $(M+\kappa_n F)du + (N+\kappa_n G)dv$ $= 0$
(3.42)	$\begin{vmatrix} L-\kappa_n E & M-\kappa_n F \\ M-\kappa_n F & N-\kappa_n G \end{vmatrix} = 0$	$\begin{vmatrix} L+\kappa_n E & M+\kappa_n F \\ M+\kappa_n F & N+\kappa_n G \end{vmatrix} = 0$
(3.43)	$(EG-F^2)\kappa_n^2$ $-(EN+GL-2FM)\kappa_n$ $+(LN-M^2) = 0$	$(EG-F^2)\kappa_n^2$ $+(EN+GL-2FM)\kappa_n$ $+(LN-M^2) = 0$
(3.47)	$H = \frac{EN+GL-2FM}{2(EG-F^2)}$	$H = \frac{2FM-EN-GL}{2(EG-F^2)}$
(3.51)	$\lambda = -\frac{M-\kappa_n F}{N-\kappa_n G}$	$\lambda = -\frac{M+\kappa_n F}{N+\kappa_n G}$
(3.52)	$\lambda = -\frac{L-\kappa_n E}{M-\kappa_n F}$	$\lambda = -\frac{L+\kappa_n E}{M+\kappa_n F}$
(3.67)	$H =$ $\frac{(1+h_x^2)h_{yy}-2h_xh_yh_{xy}+(1+h_y^2)h_{xx}}{2(1+h_x^2+h_y^2)^{3/2}}$	$H =$ $\frac{2h_xh_yh_{xy}-(1+h_x^2)h_{yy}-(1+h_y^2)h_{xx}}{2(1+h_x^2+h_y^2)^{3/2}}$
(3.88)	$\kappa_n = \frac{Ldu^2+Ndv^2}{Edu^2+Gdv^2}$	$\kappa_n = -\frac{Ldu^2+Ndv^2}{Edu^2+Gdv^2}$
(3.89)	$\kappa_1 = \frac{L}{E}, \qquad \kappa_2 = \frac{N}{G}$	$\kappa_1 = -\frac{L}{E}, \qquad \kappa_2 = -\frac{N}{G}$